Efficacy of measurement reversal for stochastic disturbances

David J. Starling[1,*] and Nathan S. Williams[2]
[1]*Division of Science, Pennsylvania State University, Hazleton, Pennsylvania 18202, USA*
[2]*Department of Physics, Willamette University, 900 State Street, Salem, Oregon 97301, USA*
(Received 10 April 2013; published 30 August 2013)

We analyze the statistical properties of entanglement for pairs and triples of entangled qubits subject to a random disturbance followed by restoration. After the random disturbance, which is modeled as a null-result weak measurement, the state is restored by applying a static measurement reversal. We then show that the fidelity of the resulting state, and therefore its entanglement, can be restored with a high success, despite the statistical fluctuations of the disturbance. In particular, we show that the variance of the entanglement of an ensemble of restored states is substantially reduced, despite large disturbances. We conclude with a proposed experimental implementation.

DOI: 10.1103/PhysRevA.88.024304 PACS number(s): 03.67.Bg, 03.65.Ud, 05.40.−a, 03.65.Ta

I. INTRODUCTION

Preservation and restoration of the quantum state of correlated particles are currently an active topic of research in condensed matter [1] and optical systems [2,3]. In particular, it is often desirable to increase the amount of entanglement available in a system for the purpose of quantum information, computation, or communication. There are a variety of methods used to achieve such behavior, including entanglement distillation [2], use of decoherence-free subspaces to avoid degradation altogether [4], and quantum measurement reversal [5]. Recently, the probabilistic recovery of entanglement after a disturbance has been shown in the context of entangled photons [6]. However, there are many situations in which the disturbance varies in time. Here, we explore the efficacy of quantum measurement reversal for bipartite and tripartite entanglement when one party is subjected to a stochastic disturbance. We find that using a static weak measurement, as in Ref. [6], results in high fidelity and entanglement, even for random disturbances. Furthermore, the *variance* of the entanglement for an ensemble of stochastically disturbed parties is reduced by a factor of 23 or more.

Information carriers that exhibit quantum correlations, either entanglement [7–9], squeezing [10–13], or discord [14], have found many uses in the modern world of quantum information. The protection and recovery of such resources have recently become surging topics [3,5,6,15–22]. One particular method is that of entanglement distillation, where one sifts through a large ensemble of partially entangled systems to distill a much smaller ensemble of highly entangled systems. This has been demonstrated by Kwiat *et al.* [2] for an optical system based on polarization entanglement and can even be achieved by starting with tripartite states [23].

Another technique to preserve entanglement is by avoiding decoherence altogether [4]. This is done, for example, by deterministically evolving the entangled qubits into states robust to amplitude damping and dephasing [24]. It is also possible to couple qubits to a noiseless subsystem for information storage; this has been experimentally demonstrated in the case of solid-state qubits [1].

While entanglement distillation and decoherence-free subspaces are powerful, there is another way to restore entanglement in a quantum system. Ueda and Kitagawa [18] introduced the idea of reversing a measurement, thereby *undoing* the decoherence of a quantum system. This original work quickly spawned a number of interesting results [3,5,19–22,25]. It has been shown that when an unknown initial state is disturbed by a *known* measurement, this measurement can be probabilistically reversed. The resulting final state resembles the undisturbed initial state with a high fidelity [3,19,25]. This method makes use of a single weak measurement that slightly perturbs the state and results in a small amount of information recovered. While fundamentally different, entanglement distillation and measurement reversal have been connected in a paper by Sun *et al.* [5], where they examine how entanglement decreases when the entangled qubits are locally disturbed by either amplitude damping or a null-result weak measurement. Then, knowing the strength of the disturbance, they put each qubit through a local reversal process, which in many cases will regain some of the lost entanglement.

But how does this weak measurement reversal perform if the disturbance is stochastic? To answer this question, we consider a nonlocal fixed measurement reversal in the presence of statistical disturbances. In Sec. II we set the stage for our protocol when applied to bipartite systems and derive analytic results when a stochastic disturbance is present. In Sec. III we extend the results to three parties, showing that measurement reversal is effective for larger systems. We conclude in Sec. IV by proposing an experimental implementation to correct for stochastic disturbances via a single local measurement of photon polarization.

II. BIPARTITE SYSTEMS

In the following sections, we discuss the evolution of the state of a collection of entangled qubits. We write the states in the computational $\{|0\rangle,|1\rangle\}$ basis and first consider a *pair* of qubits. The state of our system can then be represented by the joint density matrix ρ in the joint Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ of both qubits, A and B. We characterize the disturbance as a null-result weak measurement [5] with an operator defined by

$$\hat{D} = e^{-\phi_0}\hat{\Pi}_0 + e^{-\phi_1}\hat{\Pi}_1, \tag{1}$$

*starling@psu.edu

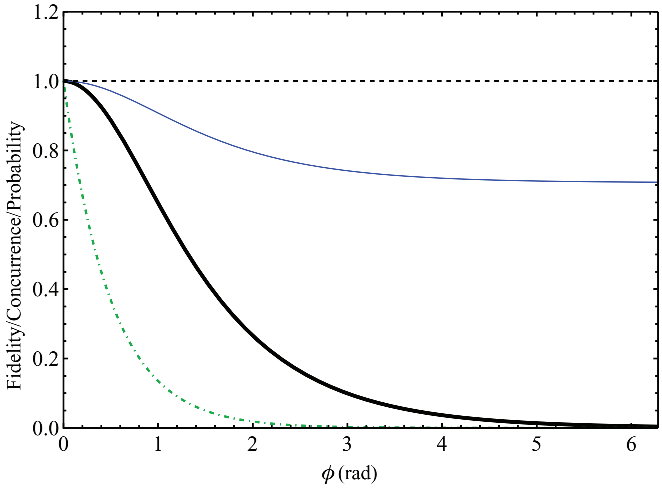

FIG. 1. (Color online) The fidelity [thin solid (blue) line] and concurrence [thick solid (black) line] of a singlet state after a disturbance of strength ϕ. The uncorrected (corrected) state is shown as a solid (dashed) line. Note that the reversal measurement has a strength that matches the disturbance, resulting in a perfectly restored state. However, the probability of correction [dash-dotted (green) line] decreases with growing disturbance.

where $\hat{\Pi}_{\{0,1\}}$ are the projectors onto $\{|0\rangle,|1\rangle\}$ and $\phi_{\{0,1\}}$ are the complex phases describing the disturbance of each basis state. The imaginary part of $\phi_{\{0,1\}}$ describes a phase shift, whereas the real part describes an attenuation (when $\mathrm{Re}[\phi_{\{0,1\}}] > 0$) or an amplification (when $\mathrm{Re}[\phi_{\{0,1\}}] < 0$). In what follows, we restrict our analysis to attenuation.

Using this disturbance operator, we can calculate the normalized state of the qubits after the disturbance in a straightforward manner,

$$\rho' = \frac{(\hat{D}\otimes\hat{\mathbb{1}})\rho(\hat{D}\otimes\hat{\mathbb{1}})^\dagger}{\mathrm{Tr}[(\hat{D}\otimes\hat{\mathbb{1}})\rho(\hat{D}\otimes\hat{\mathbb{1}})^\dagger]}, \tag{2}$$

where only qubit A has been disturbed.

For bipartite systems, we choose to evaluate the disturbance using two common measures: fidelity, defined as $\mathcal{F}(\rho,\rho') = \mathrm{Tr}[\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}]$, and concurrence $C(\rho)$, the well-known entanglement monotone. To obtain analytic results, we restrict the calculation of these two measures to the maximally entangled singlet state:

$$|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}. \tag{3}$$

Without loss of generality, we assume that only one basis state is attenuated so that $\phi_0 = \phi$ and $\phi_1 = 0$. The fidelity and concurrence can then be expressed as

$$\mathcal{F}(\rho,\rho') = \sqrt{\frac{1+\mathrm{sech}(\phi)}{2}}, \tag{4}$$

$$C(\rho') = \mathrm{sech}(\phi). \tag{5}$$

We know that the disturbance can be undone by performing a second, nonlocal measurement. That is, consider altering the state of the unaffected qubit B by performing a measurement of the same form as $\hat{D}$. We represent the measurement on B by the operator $\hat{D}'$; the new state is therefore given by

$$\rho'' = \frac{(\hat{\mathbb{1}}\otimes\hat{D}')\rho'(\hat{\mathbb{1}}\otimes\hat{D}')^\dagger}{\mathrm{Tr}[(\hat{\mathbb{1}}\otimes\hat{D}')\rho'(\hat{\mathbb{1}}\otimes\hat{D}')^\dagger]}, \tag{6}$$

where $\hat{D}'$ is characterized by the parameter $\phi'_{\{0,1\}}$; we set $\phi'_0 = \phi'$ and $\phi'_1 = 0$ to match the form of the original disturbance.

If we again consider the singlet state, (3), after the disturbance and measurement reversal $\{\phi,\phi'\}$, we obtain

$$\mathcal{F}(\rho,\rho'') = \sqrt{\frac{1+\mathrm{sech}(\phi-\phi')}{2}}, \tag{7}$$

$$C(\rho'') = \mathrm{sech}(\phi-\phi'). \tag{8}$$

Unsurprisingly, we find that the optimal reversal measurement $\hat{D}'$ matches the strength of the disturbance $\hat{D}$. The results are shown in Fig. 1, along with the matched correction (1 for all values of ϕ) and probability of success, given by

$$P = \mathrm{Tr}[(\hat{\mathbb{1}}\otimes\hat{D}')\rho'(\hat{\mathbb{1}}\otimes\hat{D}')^\dagger]. \tag{9}$$

If we know the nature and strength of the disturbance on qubit A, we can undo the effect by applying an analogous disturbance to qubit B. This feature of measurement is well known [5] and offers a robust way of restoring the entanglement in a system at the cost of reducing the transmission rate.

We now turn to the situation where the disturbance is statistical in nature by replacing ϕ with a Gaussian random variable $\widetilde{\phi}$, with average $\langle\widetilde{\phi}\rangle = \phi_a$ and variance $\sigma^2 = \langle\widetilde{\phi}^2\rangle - \langle\widetilde{\phi}\rangle^2$. We have chosen a Gaussian distribution $P(\phi)$ as the most general based on the central limit theorem. We again consider a singlet state, and the static reversal measurement is set to the *average* strength of the disturbance (namely, ϕ_a).

For an ensemble of randomly disturbed qubits we can determine the probability distribution of the fidelity and concurrence of the resulting state before and after the measurement reversal. For example, for the concurrence given by Eq. (5), we let $g(\phi) = \mathrm{sech}(\phi)$ such that $g^{-1}(C) = \mathrm{sech}^{-1}(C)$. Then the probability distribution is just

$$p(C) = P[g^{-1}(C)]\left|\frac{dg^{-1}(C)}{dC}\right|. \tag{10}$$

In this manner, the probability distribution of the fidelity and concurrence can be found before and after the measurement

TABLE I. Probability density functions for concurrence and fidelity.

Before reversal	After reversal
$p(C;\rho') = \frac{e^{-[\phi_a+\mathrm{arsech}(C)]^2/2\sigma^2}}{\sqrt{2\pi}\sigma(1-C)C}(1+e^{2\phi_a\mathrm{arsech}(C)/\sigma^2})\sqrt{\frac{1-C}{1+C}}$	$p(C;\rho'') = \frac{e^{-\mathrm{arsech}^2(C)/2\sigma^2}}{\sigma(1-C)C}\sqrt{\frac{2(1-C)}{\pi(1+C)}}$
$p(F;\rho') = \lvert 2F^2-1\rvert^{-1}\sqrt{\frac{2}{\pi\sigma^2(1-F^2)}}e^{-[\phi_a-\mathrm{arcosh}(2F^2-1)]^2/2\sigma^2}$	$p(F;\rho'') = \lvert 2F^2-1\rvert^{-1}\sqrt{\frac{8}{\pi\sigma^2(1-F^2)}}e^{-\mathrm{arcosh}^2(2F^2-1)/2\sigma^2}$

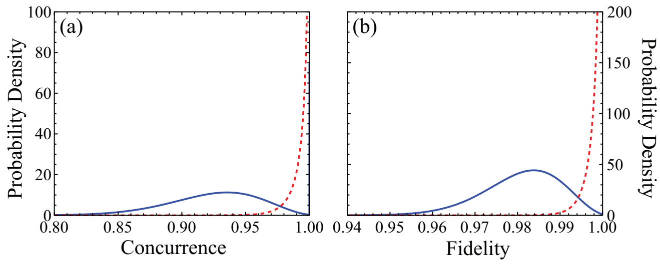

FIG. 2. (Color online) Probability distribution of the (a) concurrence and (b) fidelity of an ensemble of qubits subjected to a random disturbance before [solid (blue) line] and after [dashed (red) line] measurement reversal. The disturbance is quantified by a Gaussian random variable with mean $\phi_a = \pi/8$ and standard deviation $\sigma = \pi/30$.

reversal. The results of this analysis were checked against Monte Carlo simulation and are listed in Table I.

To give some sense of how effective this procedure is, we plot these four expressions using $\phi_a = \pi/8$ and $\sigma = \pi/30$ in Fig. 2. For this disturbance, the average concurrence improved after measurement reversal, from $\langle C\rangle = 0.924 \rightarrow 0.995$ with $\sigma_C^2 = 1.3\times10^{-3} \rightarrow 5.7\times10^{-5}$. Similarly, the average fidelity improved from $\langle \mathcal{F}\rangle = 0.981 \rightarrow 0.999$ with $\sigma_{\mathcal{F}}^2 = 8.4\times10^{-5} \rightarrow 3.6\times10^{-6}$. This represents a reduction in the spread of the concurrence and fidelity by a factor of ≈23.

In Fig. 2, we consider specific parameters for our statistical disturbance ($\phi_a = \pi/8$ and $\sigma = \pi/30$); however, the factor by which the variance is reduced depends on these parameters. We calculate the variance ratio (i.e., how much the spread of the concurrence is reduced after measurement reversal) for a variety of initial variances and plot this figure of merit in Fig. 3 as a function of the disturbance ϕ_a. We find that the ability to reduce the variance with measurement reversal *increases* as the width of the disturbance *decreases*, and for each variance

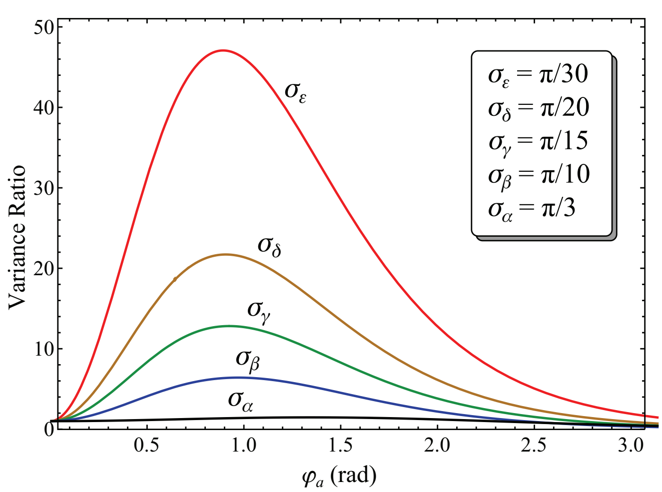

FIG. 3. (Color online) Plot of the ratio of the variances of the concurrence distributions $p(C;\rho')$ and $p(C;\rho'')$ listed in Table I as a function of the mean disturbance ϕ_a. We show the ratio for a variety of disturbance spreads σ_i ($i \in \{\alpha,\beta,\gamma,\delta,\epsilon\}$). Higher ratios indicate a larger reduction in the spread of the distribution.

there is an average displacement angle that leads to the optimal reduction in spread.

III. TRIPARTITE SYSTEMS

We now extend this analysis to systems of three parties. Due to the complexity of tripartite systems, we restrict our analysis to the common entangled GHZ state, defined as

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \tag{11}$$

in the computational basis, and $\rho = |\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|$. We disturb the first qubit of the state in the same manner as before, resulting in a new state,

$$\rho' = \frac{(\hat{D}\otimes\hat{\mathbb{1}}\otimes\hat{\mathbb{1}})\rho(\hat{D}\otimes\hat{\mathbb{1}}\otimes\hat{\mathbb{1}})^\dagger}{\mathrm{Tr}[(\hat{D}\otimes\hat{\mathbb{1}}\otimes\hat{\mathbb{1}})\rho(\hat{D}\otimes\hat{\mathbb{1}}\otimes\hat{\mathbb{1}})^\dagger]}, \tag{12}$$

where $\hat{D}$ is quantified by its strength $\phi_0 = \phi$ and $\phi_1 = 0$ as before.

In order to reverse the disturbance from $\hat{D}$, we apply a similar reversal measurement $\hat{D}'$ on any of the three qubits. In this case, $\hat{D}'$ is quantified by $\phi_0' = 0$ and $\phi_1' = \phi'$. That is, we attenuate the opposite basis state. For example, we obtain

$$\rho'' = \frac{(\hat{\mathbb{1}}\otimes\hat{D}'\otimes\hat{\mathbb{1}})\rho'(\hat{\mathbb{1}}\otimes\hat{D}'\otimes\hat{\mathbb{1}})^\dagger}{\mathrm{Tr}[(\hat{\mathbb{1}}\otimes\hat{D}'\otimes\hat{\mathbb{1}})\rho(\hat{\mathbb{1}}\otimes\hat{D}'\otimes\hat{\mathbb{1}})^\dagger]}. \tag{13}$$

Note that, had we chosen a different initial state (e.g., $|W\rangle$), the reversal measurement might also be different. The correction is, therefore, *initial state dependent*. While this feature may seem undesirable, it is common to work with highly entangled states of a specified type; therefore, we can tailor our reversal for the system at hand.

We calculate the fidelity as before and use the so-called GME concurrence [26] to quantify the entanglement, defined for tripartite pure states to be

$$C_{\mathrm{GME}}(\rho) = \min_{i\in\{1,2,3\}}\left[\sqrt{2\left(1-\mathrm{Tr}\rho_i^2\right)}\right], \tag{14}$$

where ρ_i is the reduced density matrix of the ith qubit. This has been shown to satisfy the minimum criteria for a measure of genuine multipartite entanglement [27].

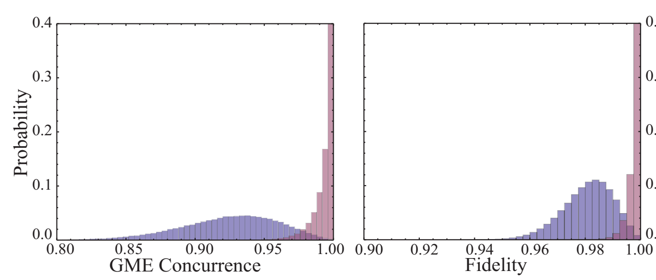

FIG. 4. (Color online) Probability distributions of the (a) GME concurrence and (b) fidelity of a GHZ state subjected to a random disturbance before [leftward (blue) bars] and after [rightward (red) bars] measurement reversal. The disturbance is quantified by a Gaussian random variable with mean $\phi_a = \pi/8$ and standard deviation $\sigma = \pi/30$.

As with the bipartite case, we find that the static measurement reversal of a randomly disturbed qubit in an entangled GHZ state improves GME concurrence and fidelity on average and reduces the spread of their values. We perform a Monte Carlo simulation with $N = 10^5$ random samplings and $\phi_a = \pi/8$ and $\sigma = \pi/30$ as before. The results are shown in Fig. 4. The average GME concurrence improved from $\langle C_{\text{GME}} \rangle = 0.924 \to 0.995$ with $\sigma_C^2 = 1.3 \times 10^{-3} \to 5.7 \times 10^{-5}$. Similarly, the average fidelity improved from $\langle \mathcal{F} \rangle = 0.981 \to 0.999$ with $\sigma_{\mathcal{F}}^2 = 8.4 \times 10^{-5} \to 3.5 \times 10^{-6}$. This represents a reduction in the spread of the GME concurrence and fidelity by a factor of ≈24, as before.

IV. CONCLUSION

There is a straightforward optical implementation to test the results in Sec. II using entangled photons from spontaneous parametric down conversion; in this case, $\{|0\rangle,|1\rangle\} \to \{|H\rangle,|V\rangle\}$ in the polarization basis. Each photon is subjected to a polarization interferometer made up a polarizing beam splitter and wave plates to control the transmission probability of each polarization. One photon obtains a random disturbance (representing dichroism) and the other photon obtains a static measurement reversal. Tomography is performed on the output photons with polarization analyzers.

We have shown that for an ensemble of randomly disturbed qubits, a static reversal measurement can improve the entanglement and fidelity *as well as their variances*. The measurement reversal can be made nonlocally; i.e., we can measure (correct) an *unperturbed* qubit in order to reverse the disturbance on one of its perturbed partners. This feature is desirable from a quantum information perspective, where highly entangled resources must be transmitted with a high fidelity over long distances or else stored for future computations. The results are valid for common two- and three-qubit entangled states. We have provided analytical results and simulations and proposed a simple experimental design for implementation.

[1] L. Viola, E. M. Fortunato, M. A. Pravia, E. Knill, R. Laflamme, and D. G. Cory, Science **293**, 2059 (2001).

[2] P. G. Kwiat, S. Barraza-Lopez, A. Stefanov, and N. Gisin, Nature **409**, 1014 (2008).

[3] Y.-S. Kim, Y.-W. Cho, Y.-S. Ra, and Y.-H. Kim, Opt. Express **17**, 11978 (2009).

[4] L.-M. Duan and G.-C. Guo, Phys. Rev. Lett. **79**, 1953 (1997).

[5] Q. Sun, M. Al-Amri, L. Davidovich, and M. S. Zubairy, Phys. Rev. A **82**, 052323 (2010).

[6] Y.-S. Kim, J.-C. Lee, O. Kwon, and Y.-H. Kim, Nat. Phys. **8**, 117 (2012).

[7] C. J. Broadbent, R. M. Camacho, R. Xin, and J. C. Howell, Phys. Rev. Lett. **100**, 133602 (2008).

[8] V. Boyer, A. M. Marino, R. C. Pooser, and P. D. Lett, Science **321**, 544 (2008).

[9] M. D. Lukin and A. Imamoğlu, Phys. Rev. Lett. **84**, 1419 (2000).

[10] C. M. Caves and B. L. Schumaker, Phys. Rev. A **31**, 3068 (1985).

[11] Y. Q. Li, P. Lynam, M. Xiao, and P. J. Edwards, Phys. Rev. Lett. **78**, 3105 (1997).

[12] J. E. Sharping, M. Fiorentino, and P. Kumar, Opt. Lett. **26**, 367 (2001).

[13] P. M. Anisimov, G. M. Raterman, A. Chiruvelli, W. N. Plick, S. D. Huver, H. Lee, and J. P. Dowling, Phys. Rev. Lett. **104**, 103602 (2010).

[14] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001).

[15] B. Misra and E. C. G. Sudarshan, J. Math. Phys. **18**, 756 (1977).

[16] W. M. Itano, D. J. Heinzen, J. J. Bollinger, and D. J. Wineland, Phys. Rev. A **41**, 2295 (1990).

[17] S. Maniscalco, F. Francica, R. L. Zaffino, N. Lo Gullo, and F. Plastina, Phys. Rev. Lett. **100**, 090503 (2008).

[18] M. Ueda and M. Kitagawa, Phys. Rev. Lett. **68**, 3424 (1992).

[19] A. N. Korotkov and A. N. Jordan, Phys. Rev. Lett. **97**, 166805 (2006).

[20] A. Imamoğlu, Phys. Rev. A **47**, R4577 (1993).

[21] A. Royer, Phys. Rev. Lett. **73**, 913 (1994).

[22] M. Koashi and M. Ueda, Phys. Rev. Lett. **82**, 2598 (1999).

[23] C. Yao, Z.-H. Ma, Z.-H. Chen, and A. Serafini, Phys. Rev. A **86**, 022312 (2012).

[24] P. G. Kwiat, A. J. Berglund, J. B. Altepeter, and A. G. White, Science **290**, 498 (2000).

[25] N. Katz, M. Neeley, M. Ansmann, R. C. Bialczak, M. Hofheinz, E. Lucero, A. O'Connell, H. Wang, A. N. Cleland, J. M. Martinis, and A. N. Korotkov, Phys. Rev. Lett. **101**, 200401 (2008).

[26] Z.-H. Ma, Z.-H. Chen, J.-L. Chen, C. Spengler, A. Gabriel, and M. Huber, Phys. Rev. A **83**, 062325 (2011).

[27] O. Gühne and M. Seevinck, New J. Phys. **12**, 053002 (2010).