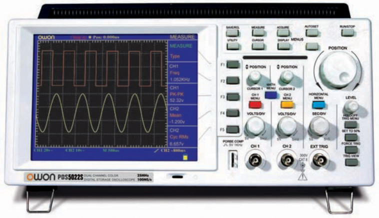
EM Oscillations



"I've got an oscillating fan at my house. The fan goes back and forth. It looks like the fan is saying 'No.' So I like to ask it questions that a fan would say 'no' to. Do you keep my hair in place? Do you keep my documents in order? Do you have 3 settings? Liar! My fan... lied to me. Now I will pull the pin up. Now you ain't sayin' [nothin']."

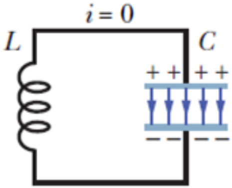
-Mitch Hedberg

David J. Starling
Penn State Hazleton
PHYS 212

What are EM Oscillations?

We have seen circuits that grow/decay exponentially:

- RC circuit: $q(t) = CV(1 - e^{-t/\tau_C})$ or $q(t) = CVe^{-t/\tau_C}$
- RL circuit: $i(t) = (V/R)(1 - e^{-t/\tau_L})$ or $i(t) = (V/R)e^{-t/\tau_L}$
- But what if we put a charged capacitor together with an inductor?



- Here, there is no energy dissipation!

What are EM Oscillations?

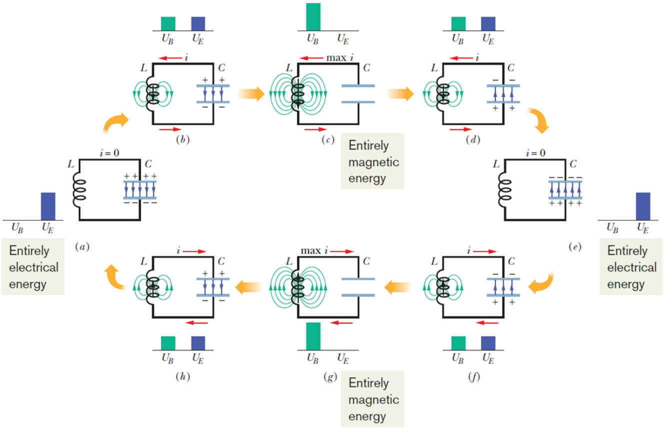
Capacitors and inductors store energy:

- $U_E = \frac{q^2}{2C}$
- $U_B = \frac{Li^2}{2}$

$$P = iV = i(Ldi/dt) \rightarrow U_B = \int Pdt = Li^2/2$$

- Initially, all the energy is stored in the capacitor and no current flows
- Eventually, the capacitor discharges and current flows
- The energy is transfered from the capacitor to the inductor... and then back!

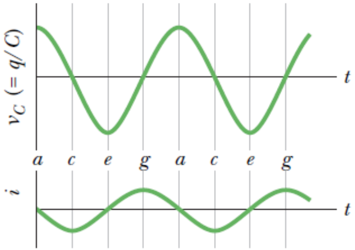
What are EM Oscillations?



What are EM Oscillations?

We can measure

- the voltage across the capacitor $V = q/C$
- the current in the circuit i
- We find:



- The voltage lags by 90°

What are EM Oscillations?

An analogy: block on a spring

- Potential energy is like the energy in a capacitor
  $U \rightarrow U_E$
- Kinetic energy is like the energy in the inductor
  $K \rightarrow U_B$

Table 31-1

Comparison of the Energy in Two Oscillating Systems

Block–Spring System		*LC* Oscillator	
Element	Energy	Element	Energy
Spring	Potential, $\frac{1}{2}kx^2$	Capacitor	Electrical, $\frac{1}{2}(1/C)q^2$
Block	Kinetic, $\frac{1}{2}mv^2$	Inductor	Magnetic, $\frac{1}{2}Li^2$
$v = dx/dt$		$i = dq/dt$	

- $\omega = \sqrt{k/m} \rightarrow \sqrt{1/LC}$

Passive Circuits

Let's look more closely at this LC circuit:

- The total energy: $U = \frac{Li^2}{2} + \frac{q^2}{2C} = \text{ const}$
- Therefore,

$$\begin{aligned}
\frac{dU}{dt} &= \frac{d}{dt}\left(\frac{Li^2}{2} + \frac{q^2}{2C}\right) \\
&= Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} \\
&= L\frac{dq}{dt}\frac{d^2q}{dt^2} + \frac{q}{C}\frac{dq}{dt} \\
&= 0 \\
\frac{d^2q}{dt^2} + \frac{q}{LC} &= 0
\end{aligned}$$

- Solution: $q(t) = Q\cos(\omega t + \phi)$, $i(t) = -I\sin(\omega t + \phi)$
- with $I = \omega Q$ and $\omega = \sqrt{1/LC}$

Passive Circuits

For an LC circuit:

$$q(t) = Q\cos(\omega t + \phi) \tag{1}$$
$$i(t) = -I\sin(\omega t + \phi) \tag{2}$$

Therefore, the energy stored is:

- $U_E = \frac{q^2}{2C} = \frac{Q^2}{2C}\cos^2(\omega t + \phi)$
- $U_B = \frac{Li^2}{2} = \frac{1}{2}LI^2\sin^2(\omega t + \phi) = \frac{Q^2}{2C}\sin^2(\omega t + \phi)$

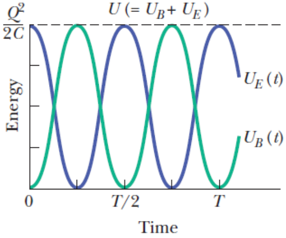
  $$\text{[note: } I^2 = (\omega Q)^2 = Q^2/LC\text{]}$$
- Adding them up, we get

  $U = U_E + U_B = \frac{Q^2}{2C}[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)] = \frac{Q^2}{2C}$

Passive Circuits

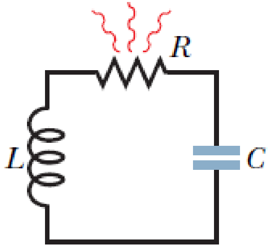
For an LC circuit:

$$\begin{aligned} q(t) &= Q\cos(\omega t + \phi) \\ i(t) &= -I\sin(\omega t + \phi) \\ U_E &= \frac{Q^2}{2C}\cos^2(\omega t + \phi) \\ U_B &= \frac{Q^2}{2C}\sin^2(\omega t + \phi) \\ U_{total} &= \frac{Q^2}{2C} \end{aligned}$$



Passive Circuits

What if we throw a resistor in the mix?



How does the previous analysis change?

- Energy is no longer constant, but decreases over time
- The power dissipated by a resistor is just $P = iV = i^2R$
- Therefore,

$$\frac{dU}{dt} = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = -i^2R \tag{3}$$

Passive Circuits

The equation for an RLC circuit:

$$
\begin{aligned}
Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} &= -i^2 R \\
Li\frac{di}{dt} + \frac{q}{C} i &= -i^2 R \\
L\frac{d^2q}{dt^2} + \frac{q}{C} + iR &= 0 \\
L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} &= 0
\end{aligned}
$$

The solution:

$$q(t) = Qe^{-Rt/2L}\cos(\omega' t + \phi)$$

Frequency:

- $\omega' = \sqrt{\omega^2 - (R/2L)^2}$
- $\omega = \sqrt{1/LC}$

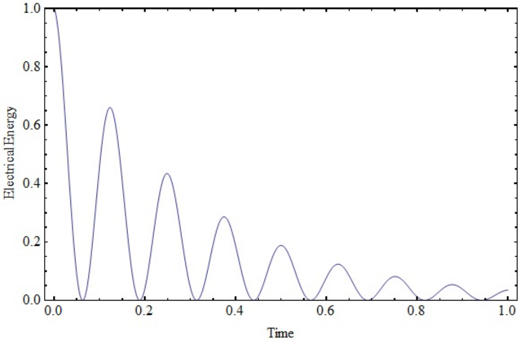
Passive Circuits

For an RLC circuit:

$$q(t) = Qe^{-Rt/2L}\cos(\omega' t + \phi)$$

$$U_E(t) = \frac{q^2}{2C} = \frac{Q^2}{2C}e^{-Rt/L}\cos^2(\omega' t + \phi)$$

The energy in the circuit dissipates over a time with a time constant $\tau_L = L/R$



Lecture Question 15.2

The current in an oscillating LC circuit is zero. Which one of the following statements is true?

(a) The charge on the capacitor is equal to zero coulombs.

(b) Charge is moving through the inductor.

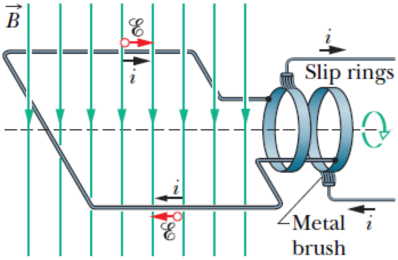
(c) The energy is equally shared between the electric and magnetic fields.

(d) The energy in the electric field is maximized.

(e) The energy in the magnetic field is maximized.

Alternating Current (AC)

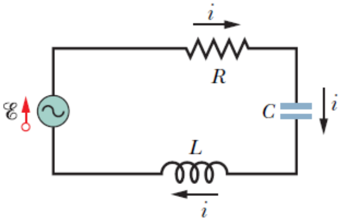
Generators create an oscillating emf:



- As the flux varies, so does the enduced emf $\mathcal{E}$
- $\mathcal{E}(t) = \mathcal{E}\sin(\omega_d t)$
- Therefore, $i(t) = I\sin(\omega_d t - \phi)$

Alternating Current (AC)

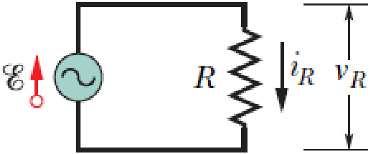
A general RLC circuit with an oscillating emf:



- The oscillations in the circuit are at frequency ω_d
- This is true even if $\omega_d \neq \omega'$
- The current moves back and forth through each element

Alternating Current (AC)

Resistive load:

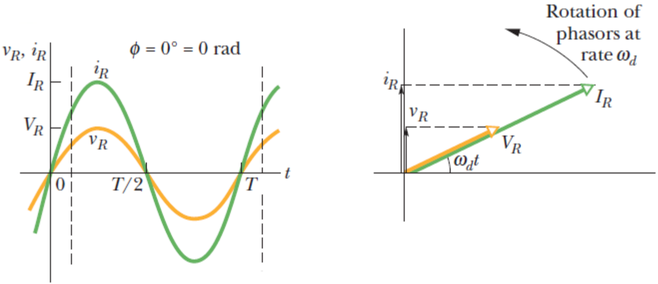


For this case,

- $\mathcal{E} - iR = 0$
- $i = \mathcal{E}/R = \mathcal{E}_m \sin(\omega_d t)/R$
- In general, we expect $i(t) = I \sin(\omega_d t - \phi)$
- For a resistive load, there is no phase delay ($\phi = 0°$)

Alternating Current (AC)

We can graph the applied voltage and the resulting current:

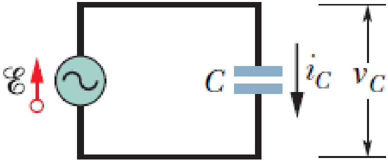


On the right, we have a **phasor diagram**

- The angle of the phasor gives the argument of the sine
- The length of the phasor is the max value of v or i
- The projection onto the y-axis gives the instantaneous v or i

Alternating Current (AC)

Capacitive load:

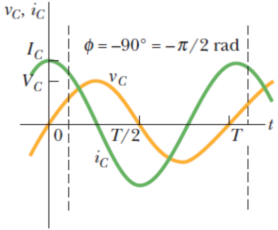


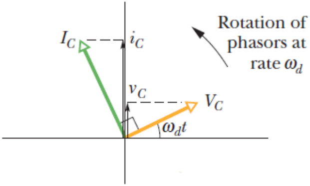
For this case,

- $q = Cv = CV\sin(\omega_d t)$
- $i = dq/dt = \omega_d CV\cos(\omega_d t) = (V/X_C)\cos(\omega_d t)$
- The **Capacitive Reactance**: $X_C = 1/\omega_d C$
- X_C has units of ohms and $V = IX_C$.

Alternating Current (AC)

We can graph the applied voltage and the resulting current:



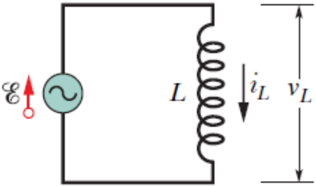


On the right, we have a **phasor diagram**

- Notice how the current leads the voltage by 90°
- $q = CV \sin(\omega_d t)$
- $i = (V/X_C) \sin(\omega_d t + 90°)$

Alternating Current (AC)

Inductive load:



For this case,

- $v = L di/dt = V \sin(\omega_d t)$
- $\frac{di}{dt} = (V/L) \sin(\omega_d t)$

Alternating Current (AC)

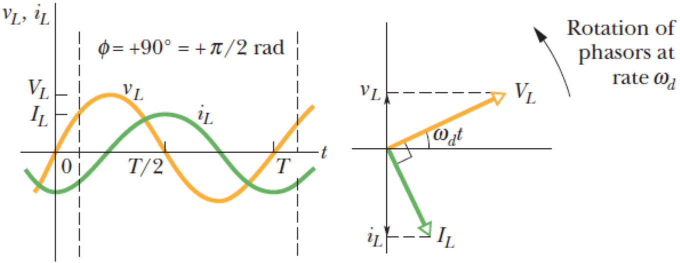
- Solving for $i(t)$,

$$\begin{aligned} di &= (V/L)\sin(\omega_d t)dt \\ i(t) &= \frac{V}{L}\int \sin(\omega_d t)dt \\ &= -\frac{V}{\omega_d L}\cos(\omega_d t) \\ &= \frac{V}{X_L}\sin(\omega_d t - 90^\circ) \end{aligned}$$

- The **Inductive Reactance**: $X_L = \omega_d L$
- X_L has units of ohms and $V = IX_L$.

Alternating Current (AC)

We can graph the applied voltage and the resulting current:



On the right, we have a **phasor diagram**

- Notice how the current lags the voltage by 90°
- $v = V \sin(\omega_d t)$
- $i = (V/X_L) \sin(\omega_d t - 90°)$

Alternating Current (AC)

Summary of Simple Circuits

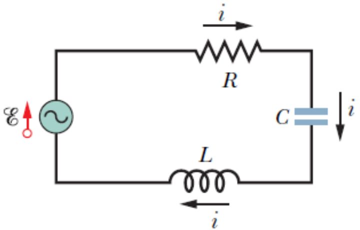
Table 31-2

Phase and Amplitude Relations for Alternating Currents and Voltages

Circuit Element	Symbol	Resistance or Reactance	Phase of the Current	Phase Constant (or Angle) ϕ	Amplitude Relation
Resistor	R	R	In phase with v_R	$0°$ ($= 0$ rad)	$V_R = I_R R$
Capacitor	C	$X_C = 1/\omega_d C$	Leads v_C by $90°$ ($= \pi/2$ rad)	$-90°$ ($= -\pi/2$ rad)	$V_C = I_C X_C$
Inductor	L	$X_L = \omega_d L$	Lags v_L by $90°$ ($= \pi/2$ rad)	$+90°$ ($= +\pi/2$ rad)	$V_L = I_L X_L$

Alternating Current (AC)

Putting them together:

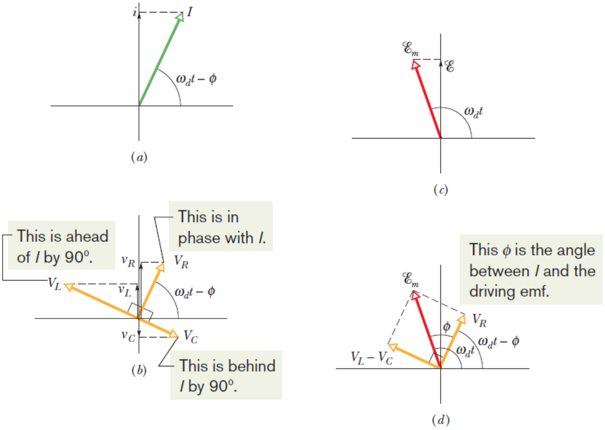


Each component operates as before:

- resistor: $i(t)$ and $v_R(t)$ are in phase
- capacitor: $i(t)$ leads $v_C(t)$ by $90°$
- inductor: $i(t)$ lags $v_L(t)$ by $90°$

Alternating Current (AC)

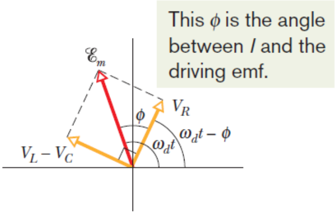
The result:



- $\mathcal{E}$ is equal to the vector sum of the three potential differences

Alternating Current (AC)

The vectors:

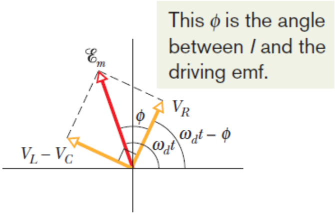


$$\begin{aligned}
\mathcal{E}_m^2 &= V_R^2 + (V_L - V_C)^2 \\
&= I^2[R^2 + (X_L - X_C)^2 \\
I &= \frac{\mathcal{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}} \\
I &= \mathcal{E}_m / Z
\end{aligned}$$

Impedence: $Z = \sqrt{R^2 + (X_L - X_C)^2}$

Alternating Current (AC)

We can now find the max current. But what is the phase between the applied voltage and the current?



$$\tan(\phi) = \frac{V_L - V_C}{V_R} = \frac{IX_L - IX_C}{IR} = \frac{X_L - X_C}{R}$$

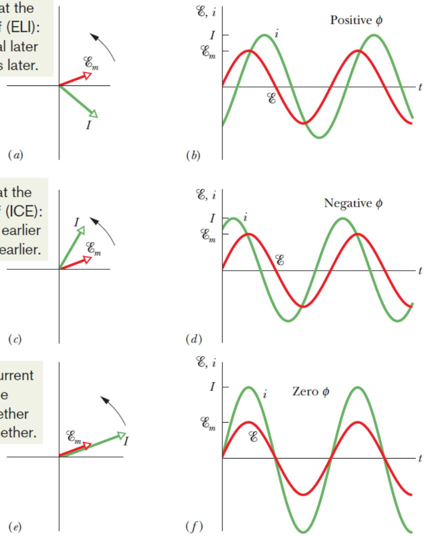
Alternating Current (AC)

$$\tan(\phi) = \frac{X_L - X_C}{R}$$

Positive ϕ means that the current lags the emf (ELI): the phasor is vertical later and the curve peaks later.

Negative ϕ means that the current leads the emf (ICE): the phasor is vertical earlier and the curve peaks earlier.

Zero ϕ means that the current and emf are in phase: the phasors are vertical together and the curves peak together.



Alternating Current (AC)

Notice that the max current depends on the impedence:

$$
\begin{aligned}
I &= \frac{\mathcal{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}} \\
&= \mathcal{E}_m / Z
\end{aligned}
$$

For a fixed R, I is max if $X_L = X_C$, so

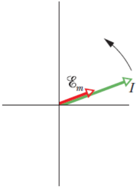
$$
\begin{aligned}
\omega_d L &= 1/\omega_d C \\
\omega_d &= 1/\sqrt{LC}
\end{aligned}
$$

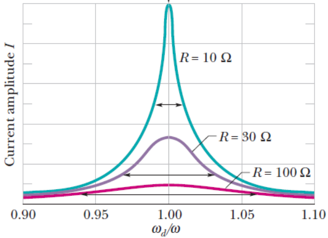
But if $X_L = X_C$, then $\phi = 0$.

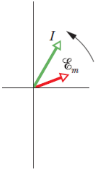
This is called resonance!

Alternating Current (AC)

Driving ω_d equal to natural ω
- high current amplitude
- circuit is in resonance
- equally capacitive and inductive
- X_c equals X_L
- current and emf in phase
- zero ϕ

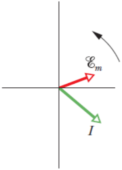






Low driving ω_d
- low current amplitude
- ICE side of the curve
- more capacitive
- X_C is greater
- current leads emf
- negative ϕ

High driving ω_d
- low current amplitude
- ELI side of the curve
- more inductive
- X_L is greater
- current lags emf
- positive ϕ



Lecture Question 15.4

An inductor circuit operates at a frequency $f = 120$ Hz. The peak voltage is 120 V and the peak current through the inductor is 2.0 A. What is the inductance of the inductor in the circuit?

(a) 0.040 H

(b) 0.080 H

(c) 0.160 H

(d) 0.320 H

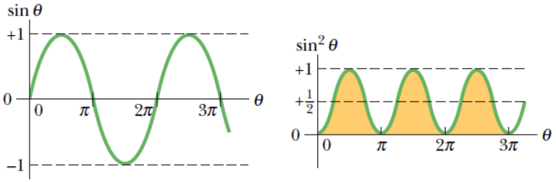
(e) 0.640 H

Power

The power dissipated in the RLC circuit leaves through the resistor only:

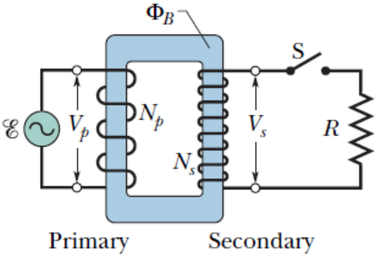
$$
\begin{aligned}
P &= i^2 R \\
&= [I \sin(\omega_d t - \phi)]^2 R \\
&= I^2 R \sin^2(\omega_d t - \phi)
\end{aligned}
$$

- The average power is just $P_{avg} = I^2 R/2$
- Define: $I_{rms} = I/\sqrt{2}$ (root-mean-squared)
- This gives $P_{avg} = I^2_{rms} R$



Power

A transformer changes one AC voltage to another AC voltage:

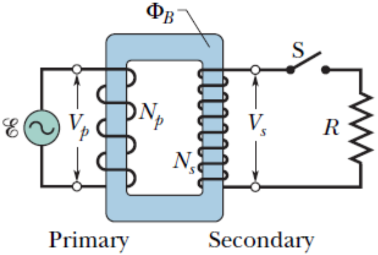


- A primary coil is wrapped around an iron core
- An emf is induced in a secondary coil also on the core
- The relationship between the two voltages is

$$V_s = V_p \frac{N_s}{N_p}$$

Power

$$V_s = V_p \frac{N_s}{N_p}$$



- Step-up transformer: $N_s > N_p$
- Step-down transformer: $N_p > N_s$
- Energy is conserved: $I_p V_p = I_s V_s \rightarrow I_s = I_p N_p / N_s$