Chapter 16 - Maxwell's Equations

Courtesy A.K. Geim, University of Manchester, UK

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PHYS 214

Gauss's Law Again!

Gauss's Law relates point charges to the value of the electric field.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$

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But what about magnetic flux?

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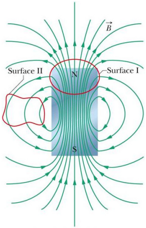
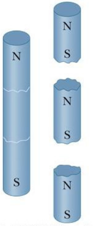
There are no magnetic monopoles!





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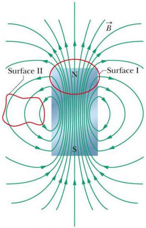

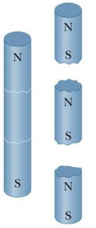

How is this different from Induction?

Gauss's Law Again!

These two equations make up half of Maxwell's Equations.

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$

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$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$

$$\oint \vec{B} \cdot d\vec{A} = 0$$

Using the gradient vector ($\vec{\nabla}$) we can differentiate both sides:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

Induced Electric Fields

We know that a changing mangetic field creates an electric field.

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$

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$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Induced Magnetic Fields

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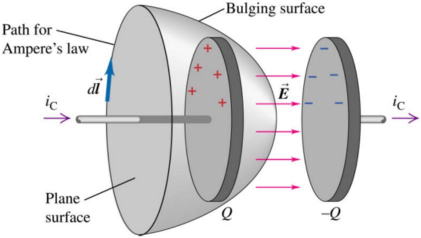
Unfortunately, it is incomplete!

Induced Magnetic Fields

Let's apply Ampere's Law for charging a capacitor with a straight wire.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc}$$



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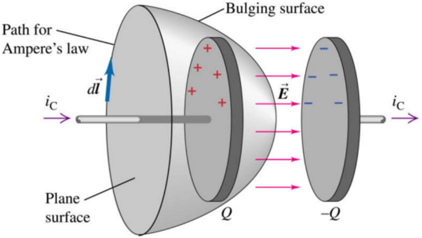

Depending on the surface, the enclosed current is different!

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Ampere's Law needs to be modified to include electric fields (as well as currents).

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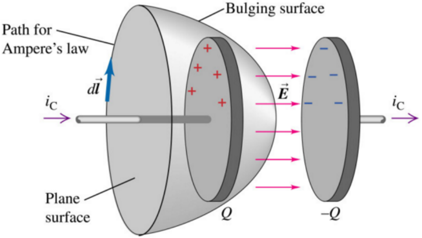

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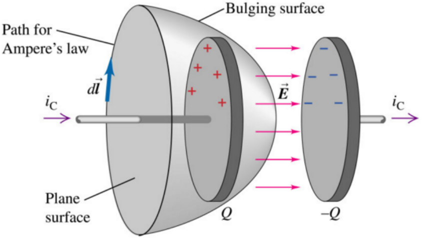

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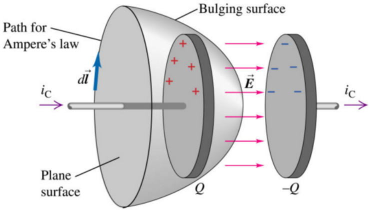

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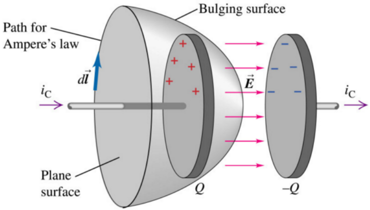

One surface has a current through it (RHS: $\mu_0 i_{enc}$) but no electric flux. The other surface has an electric flux ($\vec{E} \cdot \vec{A}$) but no current.

Induced Magnetic Fields

Regardless of which surface we use, we should get the same thing.

$$\begin{aligned}\Phi_E &= \vec{E} \cdot \vec{A} \\ &= \left(\frac{q}{\epsilon_0 A}\hat{i}\right) \cdot (A\hat{i}) \\ &= q/\epsilon_0\end{aligned}$$

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To equate with current, let's differentiate both sides:

$$\begin{aligned}\frac{d\Phi_E}{dt} &= i_{enc}/\epsilon_0 \\ \mu_0\epsilon_0\frac{d\Phi_E}{dt} &= \mu_0 i_{enc}\end{aligned}$$

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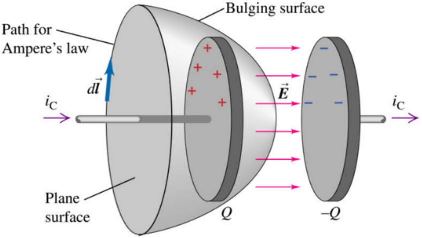
Notice how the LHS is the same as the RHS of Ampere's Law!

Induced Magnetic Fields

If we add both terms, this "Ampere-Maxwell" law is now valid for any situation.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$



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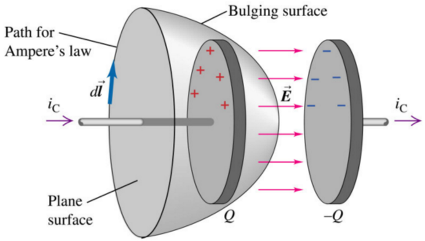

Although derived with a capacitor, this is general.

Induced Magnetic Fields

The fourth Maxwell's Equation is therefore:

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(note: we use Stoke's Theorem to relate line integrals to surface integrals)

Induced Magnetic Fields

We sometimes refer to $\epsilon_0 \frac{d\Phi_E}{dt}$ *as displacement current* $i_{d,enc}$.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \left(i_{enc} + i_{d,enc}\right)$$

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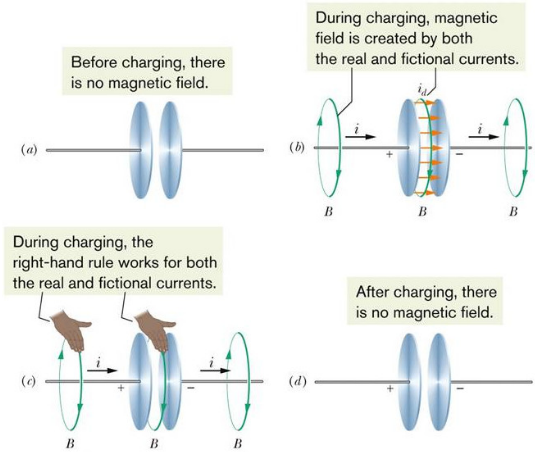
Although compact, this form is less helpful.

Let's apply our new Law!

Induced Magnetic Fields

Magnetic field from charging a capacitor.



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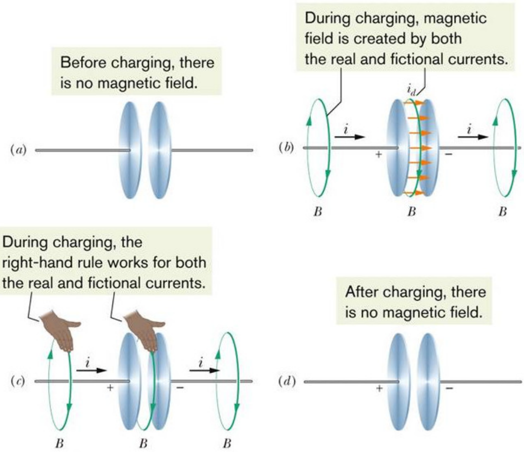
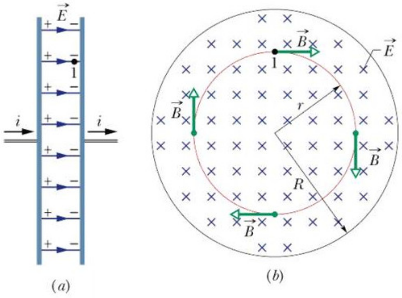


The displacement current helps us use the right hand rule.

Induced Magnetic Fields

To solve for the magnetic field between charging capacitor places, we must use the Ampere-Maxwell law.



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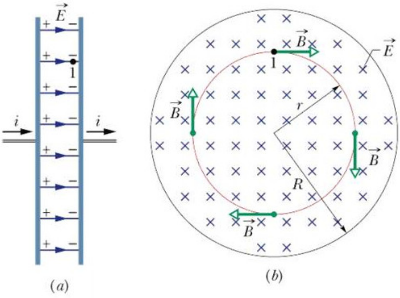

Inside, we look at a loop of fixed (arbitrary) radius *r* and calculate both sides of the Ampere-Maxwell law.

Induced Magnetic Fields

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B &= \frac{1}{2\pi r} \mu_0 \epsilon_0 \frac{d}{dt} (qA_{int} / \epsilon_0 A_{cap})
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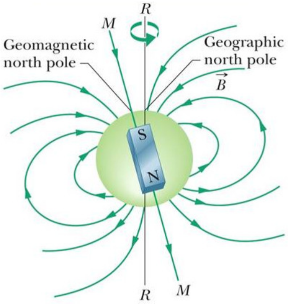
B &= \frac{\mu_0 i}{2\pi r} \frac{\pi r^2}{\pi R^2} \\
B &= \frac{\mu_0 i}{2\pi R^2} r
\end{aligned}
$$

Magnets

The Earth has its own magnetic field which can be appoximated by a bar magnet at its core.



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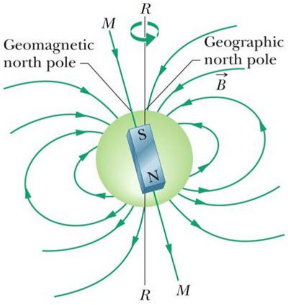
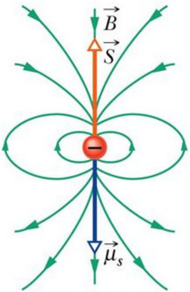


The field at the surface has a declination (from longitude) and an inclination from horizontal.

Magnets

Magnetic fields come from the motion of charges, but also from the **intrinsic property** *of charges called "spin."*



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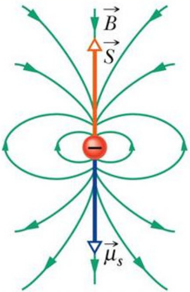

The spin $\vec{S}$ and magnetic dipole momentum $\vec{\mu}_s$ are related:

$$\vec{\mu}_s = -\frac{e}{m}\vec{S}$$

Magnets

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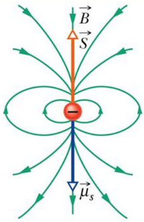
The dipole moment is therefore:

$$\mu_{s,z} = \pm\frac{e\hbar}{2m} = \pm\mu_B$$

($\mu_B = 9.27 \times 10^{-24}$ J/T is the Bohr magneton

Magnets

Like other dipoles, electrons have potential energy in an external magnetic field.



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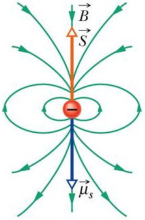
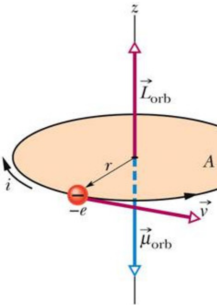


$$U_s = -\vec{\mu}_s \cdot \vec{B}_{ext} = -\mu_{s,z} B_{ext} = -\mu_B B_{ext}$$

(assuming $\vec{B}$ points in z-direction)

Magnets

As electrons move around nuclei, their motion also causes an "orbital" magnetic dipole moment.



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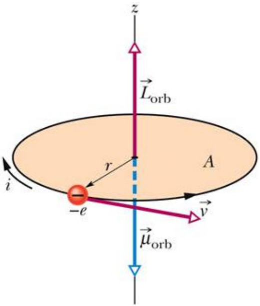

The orbital angular momentum $\vec{L}_{orb}$ and magnetic dipole momentum $\vec{\mu}_{orb}$ are related:

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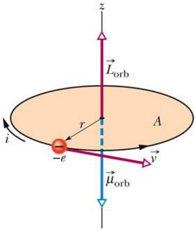

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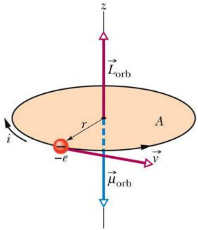

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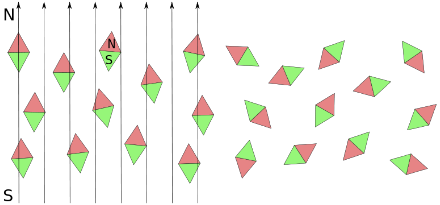
Diamagnetism is typically weak and can be found in all materials if not overwhelmed by other effects.

Magnets

Paramagnetic materials have no net magnetic field, but move toward stronger applied magnetic fields (rather than away).



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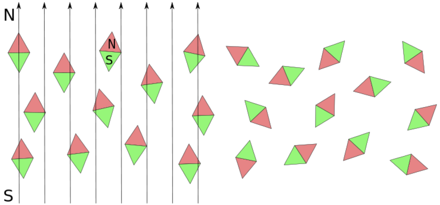
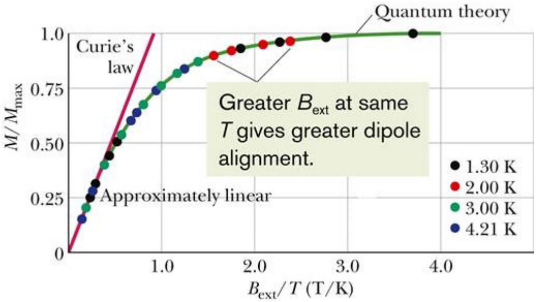


When the field is removed, the object reverts to its original non-magnetic state.

Magnets

When a material is exposed to an external magnetic field, only a portion of the magnetic dipoles rotate into alignment.



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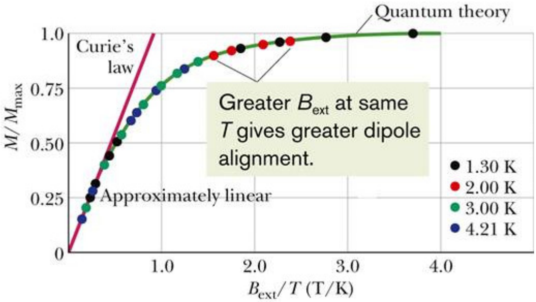
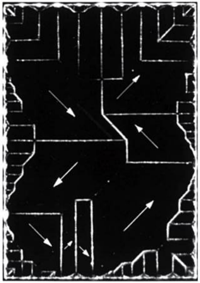


The magnetization $M = \mu_{total}/V$ is temperature dependent, with $M_{max} = N\mu/V$.

Magnets

Ferromagnetic materials are similar to paramagnetic materials, but retain their magnetic field.



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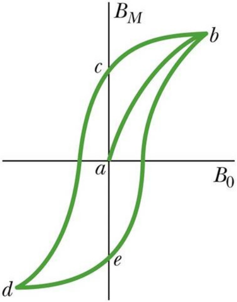

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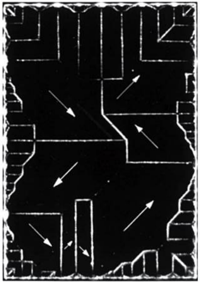

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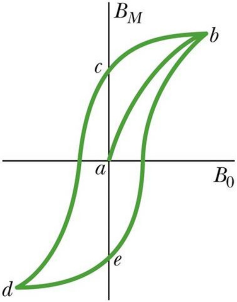

Magnetic domains form, and they persist even after the external field is removed.