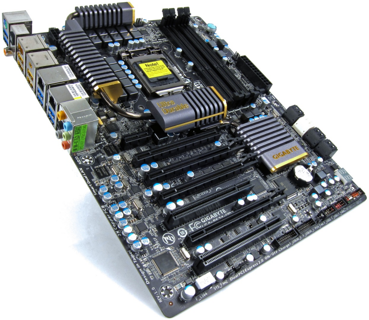
Capacitance



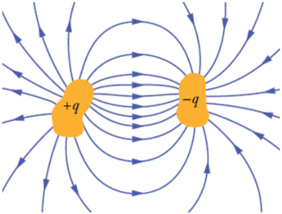
"... when I came to I had a revelation! A vision! A picture in my head! A picture of this! This is what makes time travel possible: the flux **capacitor**!"

- *Doc Brown*
Back to the Future

David J. Starling
Penn State Hazleton
PHYS 212

Capacitance

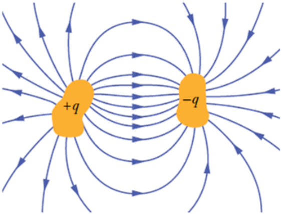
Two conductors with charges $\pm q$:



- We call this arrangement a **capacitor**
- It has charge q (even though $q_{net} = 0$)
- We can find the potential V between the two

Capacitance

Is there a simple relationship between the potential V and the charge q on a capacitor?



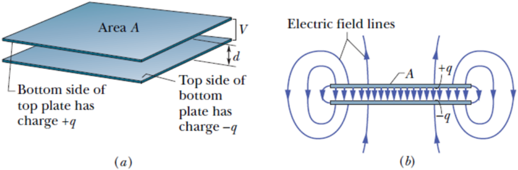
- If we increase q, V also increases
- In general:

$$q = CV \text{ or } V = q/C \text{ or } C = q/V \tag{1}$$

- C is the *capacitance* of the *capacitor*
- Units of C: the **farad**, 1 F = 1 C/V

Capacitance

Let's look at a common type of capacitor: two parallel plates.



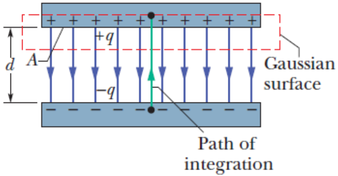
- This capacitor holds a charge q
- The field is nearly zero outside
- How can we find its capacitance?

Capacitance

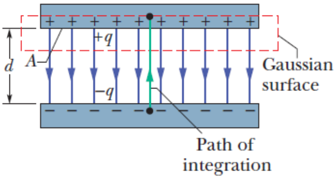
In general, to determine C:

- 1. Using Gauss's Law, find $\vec{E}$
- 2. With $\vec{E}$, we can find V
- 3. Then take the ratio $C = q/V$

To find $\vec{E}$, let's use Gauss's Law:



Capacitance

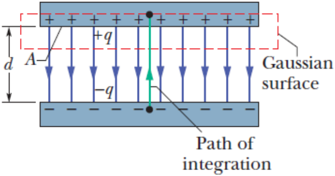


$$\begin{aligned}\Phi &= q_{enc}/\epsilon_0 \\ \oint \vec{E}\cdot d\vec{A} &= q/\epsilon_0 \\ EA &= q/\epsilon_0 \\ E &= q/A\epsilon_0\end{aligned}$$

This electric field is constant, so

$$\begin{aligned}V &= -\int_i^f \vec{E}\cdot d\vec{s} \\ &= -\int_-^+ (-Eds) = \left(\frac{q}{A\epsilon_0}\right) d\end{aligned}$$

Capacitance



Putting this together with the definition of C,

$$C = q/V = \frac{q}{qd/A\epsilon_0} = \frac{\epsilon_0 A}{d} \tag{2}$$

(capacitance for parallel plates)

Lecture Question 8.1

In calculating the electric field of two closely spaced conducting plates, it is frequently assumed that the area of the plates is larger than the distance between the plates. Why?

(a) The capacitance is too small to calculate if the plates are too far apart.

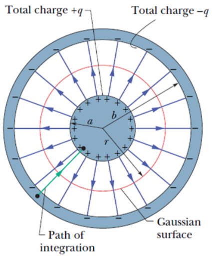
(b) The electric field near the edges of the plates is not uniform.

(c) The charge would otherwise be too small to generate a significant electric field.

(d) Gauss's law would not otherwise apply.

Capacitance

There are other common shapes for capacitors! Here are two cylinders:

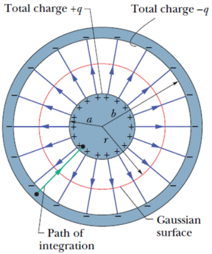


Let's find $C = q/V$.

- From Gauss's Law, $EA = q/\epsilon_0$
- For the cylinder, $A = 2\pi rL$
- $E = \frac{1}{2\pi\epsilon_0}\frac{q}{rL}$

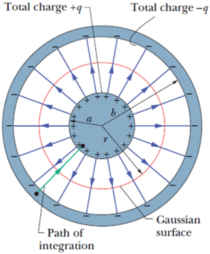
Capacitance

The E-field is no longer constant! $E = \frac{1}{2\pi\epsilon_0}\frac{q}{rL}$



$$
\begin{aligned}
V &= -\int_{-}^{+}(-Eds) \\
&= -\frac{q}{2\pi\epsilon_0 L}\int_b^a \frac{dr}{r} \quad (\text{note, } ds = -dr) \\
&= -\frac{q}{2\pi\epsilon_0 L}\ln(a/b) = \frac{q}{2\pi\epsilon_0 L}\ln(b/a)
\end{aligned}
$$

Capacitance



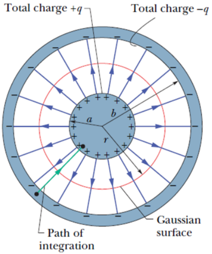
Finally, we take the ratio to get:

$$C = q/V = 2\pi\epsilon_0 \frac{L}{\ln(b/a)} \tag{3}$$

(capacitance for cylindrical plates)

Capacitance

The last example is two concentric spheres:

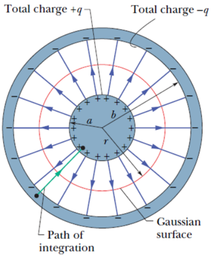


Let's find $C = q/V$.

- From Gauss's Law, $EA = q/\epsilon_0$
- For the sphere, $A = 4\pi r^2$
- $E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$ (duh)

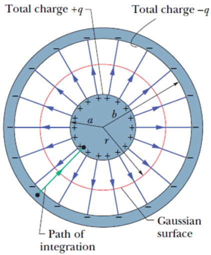
Capacitance

Again, the E-field is not constant. $E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$



$$
\begin{aligned}
V &= -\int_{-}^{+}(-Eds) \\
&= -\frac{q}{4\pi\epsilon_0}\int_b^a \frac{dr}{r^2} \quad (\text{note, } ds = -dr) \\
&= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{q}{4\pi\epsilon_0}\left(\frac{b-a}{ab}\right)
\end{aligned}
$$

Capacitance



Finally, we take the ratio to get:

$$C = q/V = 4\pi\epsilon_0 \frac{ab}{b-a} \tag{4}$$

(capacitance for spherical plates)

Capacitance

What is the capacitance of a single sphere?

- We know the potential of a sphere, relative to infinity, is $V = q/4\pi\epsilon_0 r$.
- This gives $C = q/V = 4\pi\epsilon_0 r$
- Does this agree with:

$$\begin{aligned} C &\stackrel{?}{=} \lim_{b\to\infty}\left(4\pi\epsilon_0 \frac{ab}{b-a}\right) \\ &= 4\pi\epsilon_0 \lim_{b\to\infty}\left(\frac{a}{1-a/b}\right) \\ &= 4\pi\epsilon_0 \frac{a}{1} \\ &= 4\pi\epsilon_0 r \end{aligned}$$

Lecture Question 8.2

A parallel plate capacitor with plates of area A and plate separation d is charged to a potential of V. If the capacitor is then isolated and its plate separation is increased to $2d$, what is the potential difference between the plates?

(a) 4V

(b) 2V

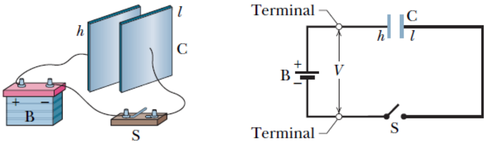
(c) V

(d) 0.5V

(e) 0.25V

Parallel Capacitance

In practice, we must "charge up" a capacitor:



(*a*) (*b*)

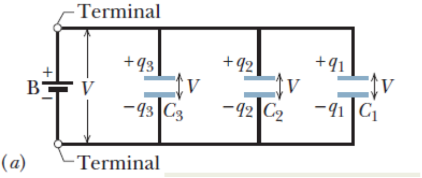
- Initially, the plates are uncharged
- The battery pushes charge with an electric field
- Electrons pile up on the negative plate (l in diagram)
- Eventually, the charge is given by $q = CV$.

Parallel Capacitance

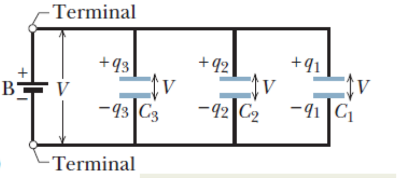
There are two common arrangements in circuits:

- Parallel — the *voltage* is the same across each capacitor
- Series — the *charge* is the same on each capacitor

Let's first look at a parallel circuit:



Parallel Capacitance



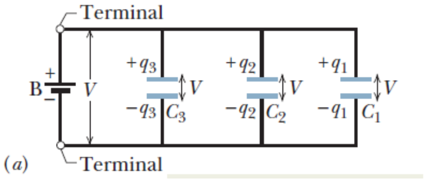
(a)

In a parallel circuit

- One plate of each capacitor is connected to a common conductor
- The other plate of each capacitor is connected to a common conductor
- This means that the voltage is the same for each capacitor!

$$q_1 = C_1 V, \; q_2 = C_2 V, \; q_3 = C_3 V$$

Parallel Capacitance



In a parallel circuit, if we add up all the charges, we find:

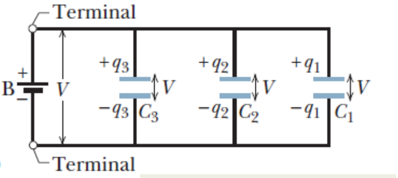
$$\begin{aligned} q &= q_1 + q_2 + q_3 \\ &= C_1V + C_2V + C_3V \\ &= (C_1 + C_2 + C_3)V \\ q &= C_{eq}V \end{aligned}$$

Capacitors in parallel have an *equivalent* capacitance of

$$C_{eq} = C_1 + C_2 + C_3 \tag{5}$$

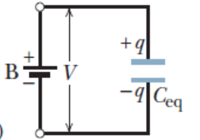
Parallel Capacitance

The parallel circuit:



(a)

Can actually be redrawn as this:



(b)

With:

$$q = C_{eq}V \text{ and } C_{eq} = C_1 + C_2 + C_3 \quad (6)$$

Lecture Question 8.3

Two capacitors A and B are connected in parallel for a long time and $C_A = 2C_B$. How does the charge on capacitor A compare to that on B?

(a) $q_A = q_B/4$

(b) $q_A = q_B/2$

(c) $q_A = q_B$

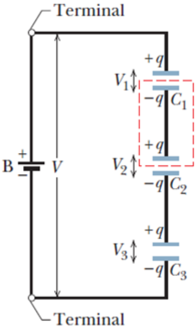
(d) $q_A = 2q_B$

(e) $q_A = 4q_B$

Series Capacitance

In a *series* circuit

- The *charge* on each capacitor has to be the same!



(*a*)

Series Capacitance

In a *series* circuit

- The potential change V from the battery is fixed
- Therefore,

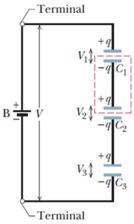
$$
\begin{aligned}
V &= V_1 + V_2 + V_3 \\
&= q/C_1 + q/C_2 + q/C_3 \\
&= q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right) \\
V &= q/C_{eq}
\end{aligned}
$$

Capacitors in series have an *equivalent* capacitance of

$$1/C_{eq} = 1/C_1 + 1/C_2 + 1/C_3 \tag{7}$$

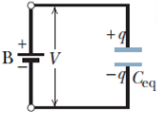
Series Capacitance

The series circuit:



(*a*)

Can actually be redrawn as this:



(*b*)

With: $q = C_{eq}V$ and $1/C_{eq} = 1/C_1 + 1/C_2 + 1/C_3$

Series Capacitance

Summary:

Capacitors in parallel have an *equivalent* capacitance of

$$C_{eq} = \sum_i C_i = C_1 + C_2 + \ldots \tag{8}$$

Capacitors in series have an *equivalent* capacitance of

$$\frac{1}{C_{eq}} = \sum_i \frac{1}{C_i} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots \tag{9}$$

Lecture Question 8.4

Two capacitors A and B are connected in series for a long time and $C_A = 2C_B$. How does the charge on capacitor A compare to that on B?

(a) $q_A = q_B/4$

(b) $q_A = q_B/2$

(c) $q_A = q_B$

(d) $q_A = 2q_B$

(e) $q_A = 4q_B$

Stored Potential Energy

Recall that potential and potential *energy* are related:

- $U = qV$
- Let's move some charge from one capacitor plate to another:

$$dU = V'dq' = (q'/C)dq'$$

- The total energy stored in a charged capacitor is just:

$$U = \int_0^q \frac{q'}{C} dq' = \frac{q^2}{2C}$$

- Or, written as

$$U = \frac{1}{2}CV^2 \tag{10}$$

Stored Potential Energy

We can think of this energy as being stored in the electric field.

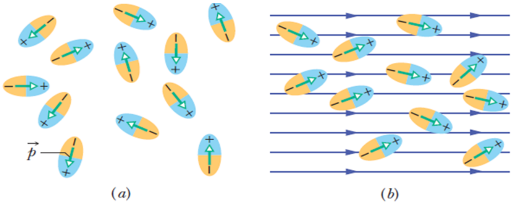
Consider two parallel plates:

- Energy *density*: $u = U/(Ad) = CV^2/(2Ad)$ (energy per volume)
- Replace C and V
- $C = \epsilon_0 A/d$
- $V = Ed$ (true for constant electric fields)
- The result:

$$u = \frac{1}{2}\epsilon_0 E^2 \tag{11}$$

Dielectric

We often think about electric fields in air, or vacuum. What happens if we fill space with a "dielectric" material?



- The molecules align with the electric field
- This *reduces* the electric field in this region
- When this happens we replace ϵ_0 with $\epsilon = \kappa\epsilon_0$

Dielectric

The constant κ varies from material to material:

Table 25-1

Some Properties of Dielectrics[a]

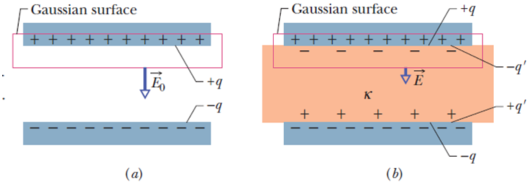
Material	Dielectric Constant κ	Dielectric Strength (kV/mm)
Air (1 atm)	1.00054	3
Polystyrene	2.6	24
Paper	3.5	16
Transformer oil	4.5	
Pyrex	4.7	14
Ruby mica	5.4	
Porcelain	6.5	
Silicon	12	
Germanium	16	
Ethanol	25	
Water (20°C)	80.4	
Water (25°C)	78.5	
Titania ceramic	130	
Strontium titanate	310	8

For a vacuum, κ = unity.

[a]Measured at room temperature, except for the water.

Dielectric

How does this affect a capacitor?



- $C = \epsilon_0 A/d \rightarrow \kappa(\epsilon_0 A/d) = \kappa C$
- Adding a material increases the capacitance
  - For the same voltage, $q = CV$ increases
  - For the same charge, $V = q/C$ decreases
  - The energy also changes: $U = CV^2/2 = q^2/2C$