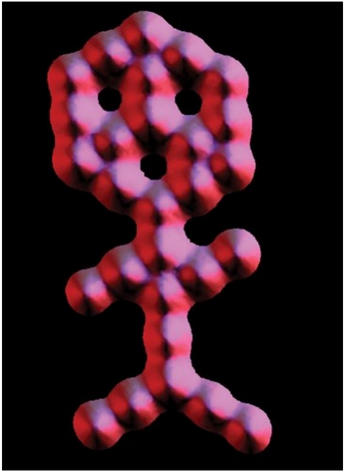
Chapter 6 - Photons and Matter Waves



"All the fifty years of conscious brooding have brought me no closer to answer the question, 'What are light quanta?' Of course today every rascal thinks he knows the answer, but he is deluding himself."

-Albert Einstein

David J. Starling
Penn State Hazleton
PHYS 214

Photon Energy

Light is not a classical wave of electric and magnetic fields. Light is composed of quanta, with energy

$$E = hf$$

where $h = 6.63 \times 10^{-34}$ J-s is the Planck constant.

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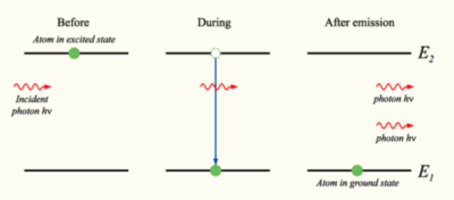

How can light be both a particle and a wave?

Photon Energy

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$$\begin{aligned}
R &= \frac{\text{Energy per Time}}{\text{Energy per Photon}} \\
&= \frac{P}{hf} \\
&= \frac{P}{hc/\lambda} \\
&= \frac{100 \times 590 \times 10^{-9}}{6.63 \times 10^{-34} \times 3 \times 10^{8}} \\
&= 2.97 \times 10^{20} \text{ photons/s.}
\end{aligned}$$

Photon Energy

How do we know that photons have discrete energy? Let's set up an experiment.

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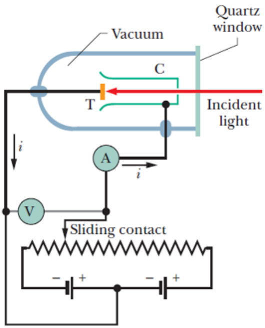

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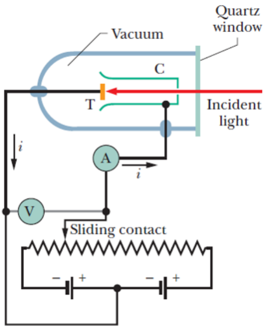
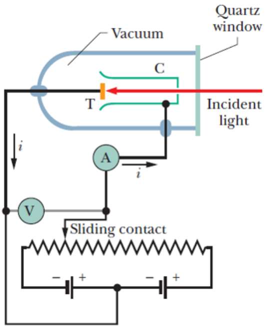


Incident light is monochromatic and knocks electrons out of the metal.

Photon Energy

We increase the voltage until no current is measured. This is the stopping potential V_{stop}.

$$K_{max} = U = eV_{stop}$$



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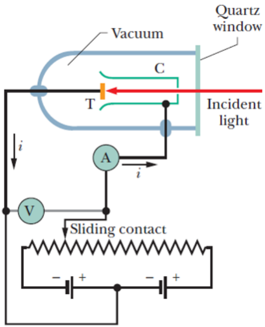
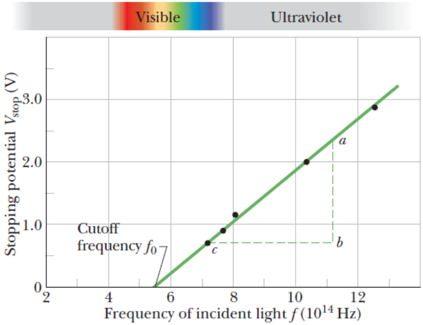


Making the light more intense *does not change the stopping potential.*

Photon Energy

Alternatively, we can change the frequency of light and measure the stopping potential for each frequency.



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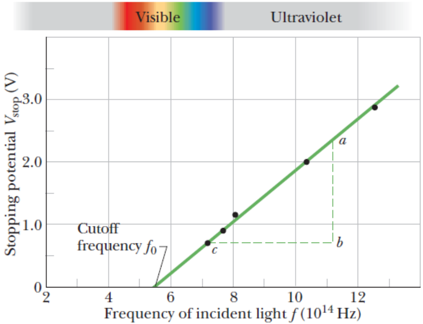
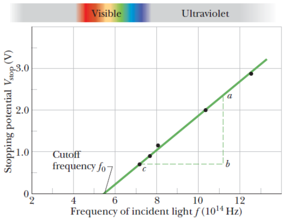


There is a minimum photon frequency (energy) under which no electrons are ejected.

Photon Energy

The resulting equation is the conservation of energy: one photon energy turns into potential and kinetic energy of an electron:

$$hf = K_{max} + \Phi$$



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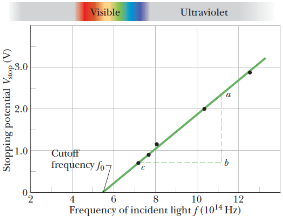
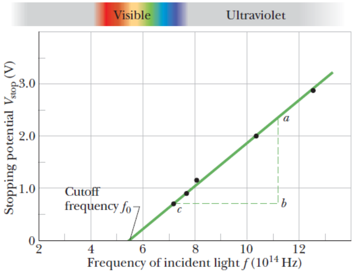


Φ is known as the "work function" and represents the energy required to pull the electron away from the material.

Photon Energy

If we replace K_{max} with eV_{stop}, we get the another form of the **photoelectric equation**.

$$V_{stop} = \frac{h}{e} f - \frac{\Phi}{e}$$



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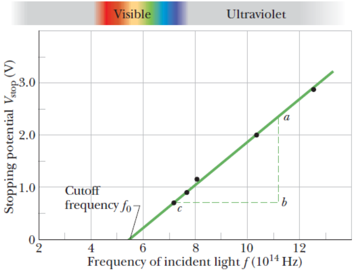


The work function depends on the material but is a constant. Therefore, we get a linear relationship.

Photon Energy

Lecture Question 6.1

Upon which one of the following parameters does the energy of a photon depend?

(a) the mass of the photon

(b) the amplitude of the electric field

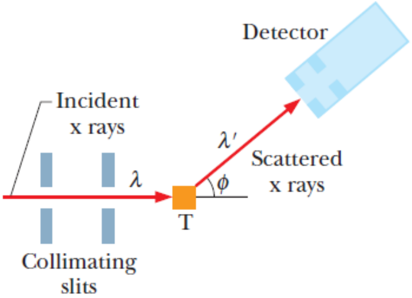
(c) the direction of the electric field

(d) the relative phase of the electromagnetic wave relative to the source that produced it

(e) the frequency of the photon

Photon Momentum

Although the photon is massless, it carries momentum $p = h/\lambda$. We know from experiments.



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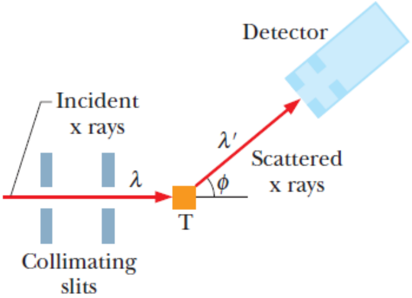
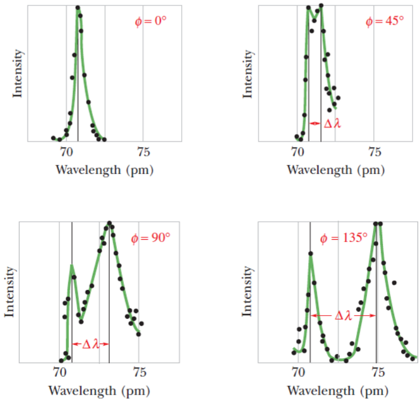


X-rays scatter off of a carbon target. The resulting angle, intensity and wavelength are measured.

Photon Momentum

Using a 71.1 pm x-ray beam, Compton measured the following "Compton Effect."



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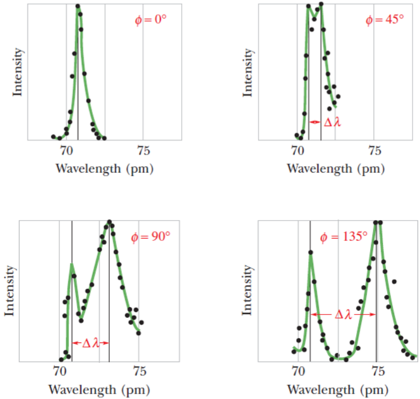
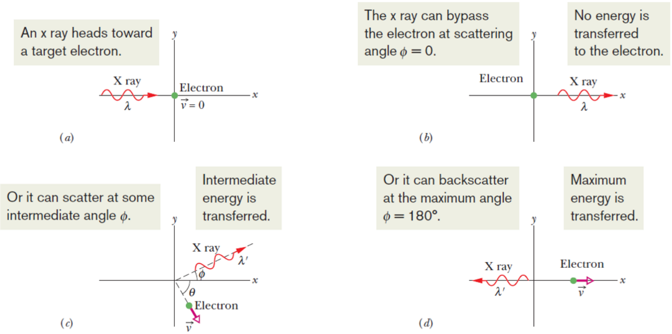


There is a shift in the energy of the x-ray photons as the scattering angle is changed.

Photon Momentum

The photon is colliding with an electron. Three things can happen.



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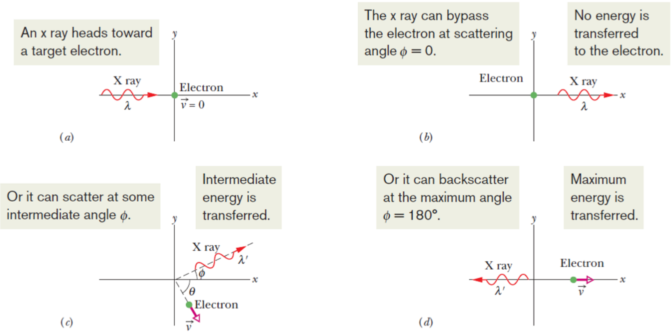

During a collision, energy must be conserved: $hf = hf' + K$.

Photon Momentum

After the collision, the electron has relativistic kinetic energy: $K = mc^2(\gamma - 1)$,

with $\gamma = 1/\sqrt{1 - (v/c)^2}$.

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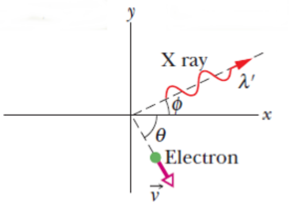
This gives:

$$\begin{aligned} hf &= hf' + mc^2(\gamma - 1) \\ \frac{h}{\lambda} &= \frac{h}{\lambda'} + mc(\gamma - 1) \text{ [I]} \end{aligned}$$

using $c = f\lambda$.

Photon Momentum

We can also consider momentum in the x- and y-directions.



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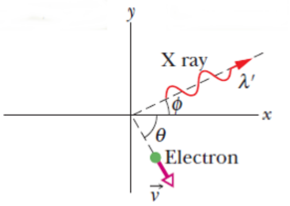


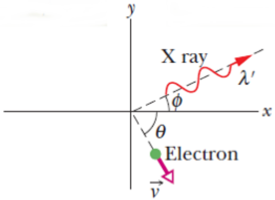
$$\frac{h}{\lambda} = \frac{h}{\lambda'}\cos(\phi) + \gamma m v \cos(\theta) \ \text{[II]}$$

$$0 = \frac{h}{\lambda'}\sin(\phi) - \gamma m v \sin(\theta) \ \text{[III]}$$

Photon Momentum

When we combine equations I, II and III to solve for $\Delta\lambda = \lambda' - \lambda$*, we get:*

$$\Delta\lambda = \frac{h}{mc}(1 - \cos\phi)$$



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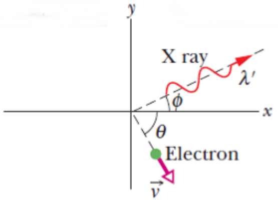
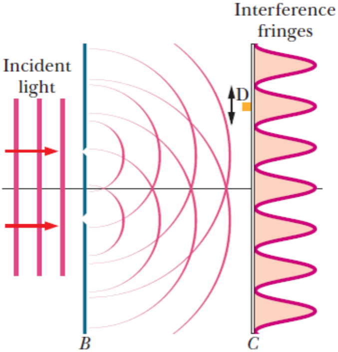


Here, we've eliminated the unknown electron properties v and θ and h/mc is the **Compton wavelength**.

Probability Waves

How do we bring together the wave nature and particle nature of light?



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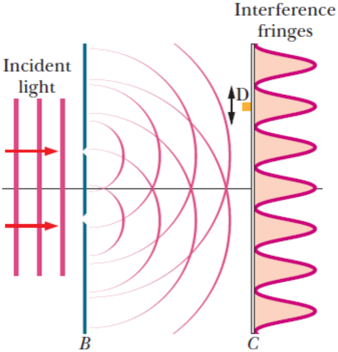
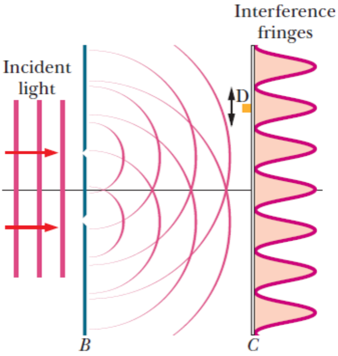


During an interference experiment, we say that the photon is spread out in a probability distribution.

Probability Waves

We associate different locations with a probability density of detecting the photon there.



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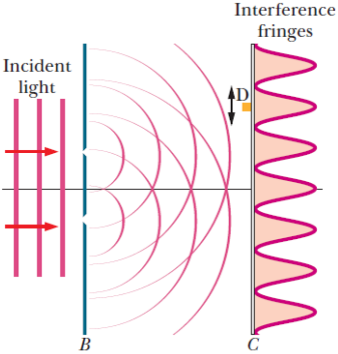
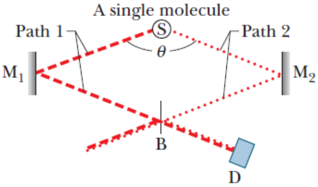


The photon is spread out like a wave while it travels, but behaves particle-like when detected.

Probability Waves

An interferometer can be thought of as a photon interfering with itself.



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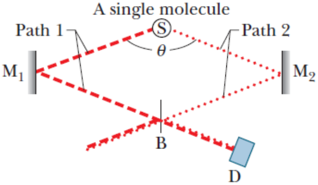
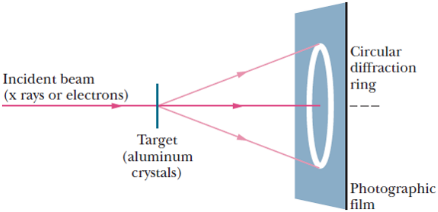


A photon is emitted, its probability wave splits into two paths, interferes with itself and is detected as a photon (or not) at D.

Probability Waves

Massive particles also behave like waves, known as matter waves, with **deBroglie wavelength**

$$\lambda = \frac{h}{p}.$$



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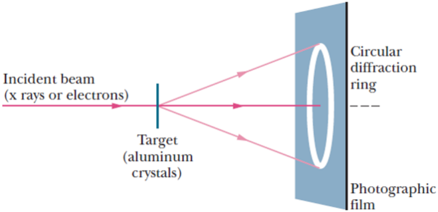

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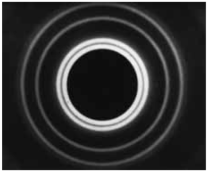

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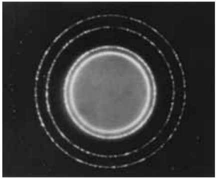
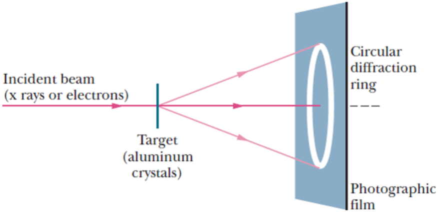


(*b*)

Probability Waves

Electrons or x-rays strike the target and then diffract like waves off of the crystalline structure.



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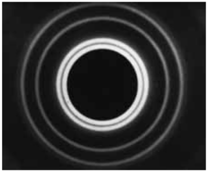

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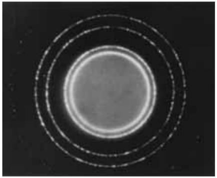


(*b*)

Probability Waves

Lecture Question 6.2

Which one of the following experiments demonstrates the wave nature of electrons?

(a) Small flashes of light can be observed when electrons strike a special screen.

(b) Electrons directed through a double slit can produce an interference pattern.

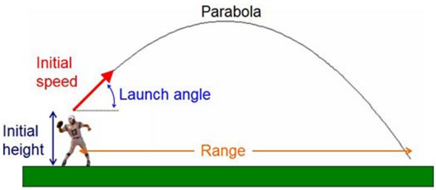
(c) The Michelson-Morley experiment confirmed the existence of electrons and their nature.

(d) In the photoelectric effect, electrons are observed to interfere with electrons in metals.

(e) Electrons are observed to interact with photons (light particles).

Schrödinger's Equation

In the past, we attempted to find the exact position $\vec{r}(t)$ of an object using Newton's Laws.



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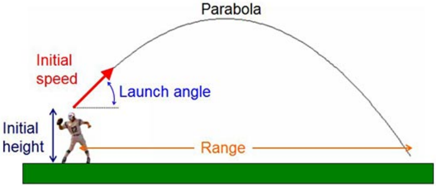
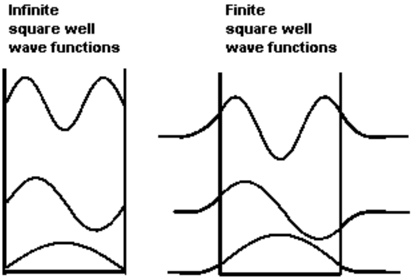


That is, we solve the differential equation $\vec{F}_{net} = m\frac{d^2\vec{r}}{dt^2}$ with initial conditions $\vec{r}_0$ and $\vec{v}_0$. The solution is

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

Schrödinger's Equation

However, in quantum mechanics, particles do not obey $\vec{F} = m\vec{a}$*, and exact positions* $\vec{r}(t)$ *do not exist. Instead, we have a probability density:* $\Psi(x, y, z, t)$



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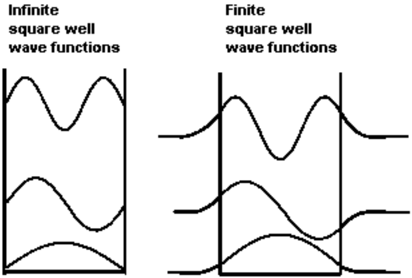

$\Psi(x, y, z, t)$ is known as the **wave function**. The probability density is given by its absolute square $p(x, y, z, t) = |\Psi|^2$.

Schrödinger's Equation

Often, the wavefunction can by simplified:

$$\Psi(x, y, z, t) = \psi(x, y, z)e^{-i\omega t}$$

where $\omega = 2\pi f$ is the angular frequency of the matter wave.

Schrödinger's Equation

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where $\omega = 2\pi f$ is the angular frequency of the matter wave.

The probability that a detector will measure a particle between a position x_1 and x_2 is given by:

$$p = \int_{x_1}^{x_2} |\psi(x)|^2 dx \rightarrow \int_V |\psi(x,y,z)|^2 dV.$$

Schrödinger's Equation

This wavefunction Ψ*, like* $\vec{r}$*, is the solution to an important differential equation.*

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$$\begin{aligned} E &= K + U \\ &= \frac{1}{2}mv^2 + U \end{aligned}$$

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Schrödinger's Equation!

$$\begin{aligned} E &= K + U \\ &= \frac{1}{2}mv^2 + U \\ &= \frac{1}{2m}(mv)^2 + U \\ &= \frac{p^2}{2m} + U \\ E\psi &= \frac{p^2}{2m}\psi + U\psi. \end{aligned}$$

Schrödinger's Equation

Let's assume that the wavefunction, which describes our wave-like electrons, is oscillatory.

$$\psi(x,t) \propto e^{i(kx-\omega t)}$$

with $k = 2\pi/\lambda = p/\hbar$ and $\hbar = h/2\pi$.

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\frac{d^2\psi}{dx^2} &= -\frac{p^2}{\hbar^2}\psi \\
-\hbar^2\frac{d^2\psi}{dx^2} &= p^2\psi \\
-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} &= \frac{p^2}{2m}\psi
\end{aligned}$$

Schrödinger's Equation

Putting this all together, we get the time-independent Schrödinger's equation.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\psi = E\psi$$

or,

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - U)\psi = 0$$

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Caveats:

(a) This is only 1-dimensional

(b) This ignores the time-oscillation

(c) The solution depends on the function $U(x)$

Schrödinger's Equation

Let's solve Schrödinger's Equation for a free particle (i.e., $U(x) = 0$*).*

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The general solution is:

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \rightarrow Ae^{ikx}(\text{right traveling})$$

Schrödinger's Equation

This solution tells us how a free particle moves in 1D. The probability density is the absolute square.

$$p = |\psi(x)|^2$$

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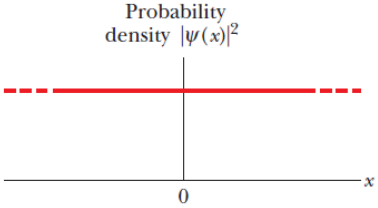
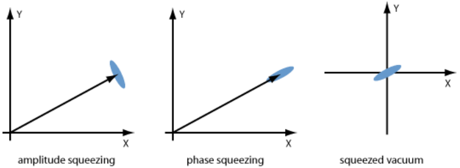


$$\begin{aligned} p &= |\psi(x)|^2 \\ &= \psi^*(x)\psi(x) \\ &= (Ae^{-ikx}) \times (Ae^{ikx}) \\ &= A^2 \\ &= \text{constant} \end{aligned}$$

Schrödinger's Equation

One consequence of this probabilistic behavior is the Heisenberg Uncertainty Principle.

$$\Delta x \cdot \Delta p_x \geq \hbar$$



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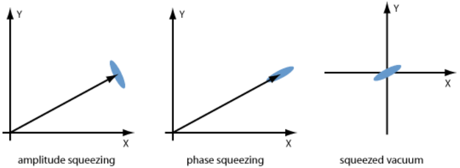


The less uncertainty in position, the more uncertainty in momentum (and *vice versa*).

Schrödinger's Equation

Lecture Question 6.3

Which one of the following statements provides the best description of the Heisenberg Uncertainty Principle?

(a) If a particle is confined to a region Δx, then its momentum is within some range Δp.

(b) If the error in measuring the position is Δx, then we can determine the error in measuring the momentum Δp.

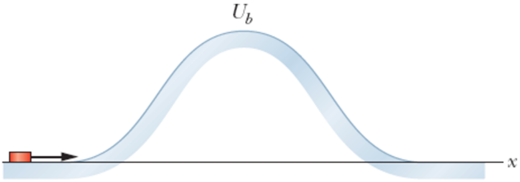
(c) If one measures the position of a particle, then the value of the momentum will change.

(d) It is not possible to be certain of any measurement.

(e) Depending on the degree of certainty in measuring the position of a particle, the degree of certainty in measuring the momentum is affected.

Tunneling

Consider a puck sliding along an icy hill where $U_b = mgh$ is the potential energy at the top.



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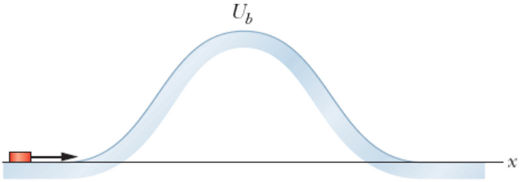
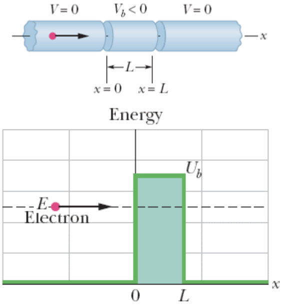


In this case, the puck needs $K > U_b$ to pass over this barrier.

Tunneling

However, in quantum mechanics, a particle can tunnel through a barrier even if it does not have enough energy to do so.



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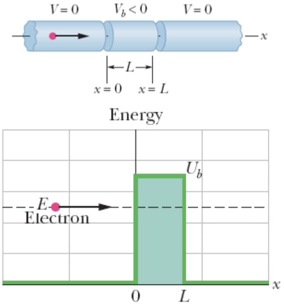
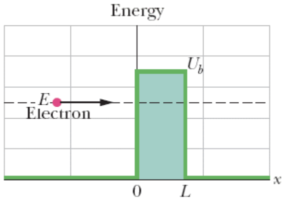


We must solve Schrödinger’s equation in order to find the tunneling probability!

Tunneling

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - U)\psi = 0$$



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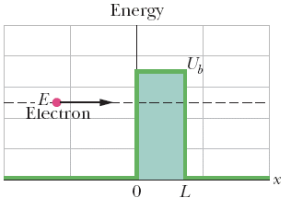

- For $x < 0$ and $x > L$, we have a free particle $U = 0$.
- For $0 < x < L$, $E < U_b = eV_b$.
- At each boundary, $\psi(x)$ must be continuous and smooth.

Tunneling

The term $E - U$ is the kinetic energy.

$$E - U = \frac{1}{2}mv^2 = \frac{p^2}{2m} = \frac{1}{2m}(k\hbar)^2$$

(note: $p = h/\lambda = k\hbar$)

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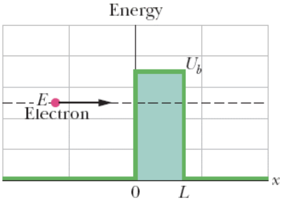
- This has the same solutions as before (e^{ikx})
- Here, $k = \sqrt{2m(E - U)}\hbar$
- The wavenumber can be imaginary if $U > E$.

Tunneling

There are three regions when an electron hits a barrier:



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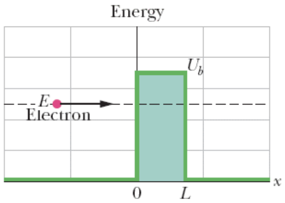

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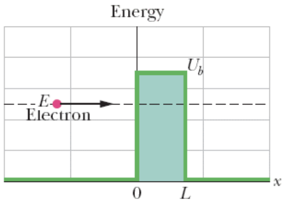

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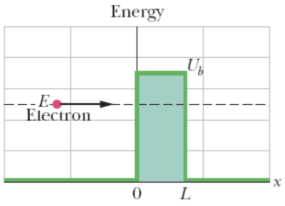

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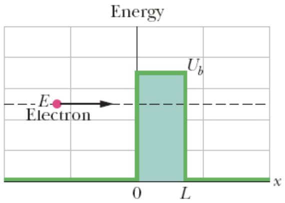

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